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Recall

Tensor Transformation Law: $T_\mu{}^\nu \rightarrow T_{\mu'}{}^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} T_\mu{}^\nu$

Partial Derivative Transformation Law: $\partial_\mu \rightarrow \partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$

Then: $$\partial_\mu T^\nu \rightarrow \partial_{\mu'} T^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \left(\frac{\partial x^{\nu'}}{\partial x^\nu} T^\nu \right)$$
$$= \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu T^\nu}_{\text{tensorial}} + \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} T^\nu \partial_\mu \left(\frac{\partial x^{\nu'}}{\partial x^\nu} \right)}_{\text{nontensorial}} \neq \text{tensor}$$

Note: $\partial_\mu \overset{\text{scalar}}{C} \rightarrow \partial_{\mu'} C = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu C$ = tensor !

Episode IV : A New Hope

We need a new derivative ! Call it $D_\mu = \partial_\mu + \underbrace{\Gamma_\mu{}^\bullet{}_\bullet}_{\text{this is the new part called a connection}}$

A few critical (and convenient) features of D_μ should be:

Connections on vectors:
a. D_μ should be a good derivative (linearity, Leibniz)
b. $D_\mu V^\nu$ should be a tensor

Extends it to connection on tensors:
c. D_μ should commute w/ contractions (summed indices)
d. D_μ should reduce to ∂_μ when acting on scalars

Specializes to Christoffel Connection:
e. Connection is torsion free
f. Metric compatibility

We will go through these one at a time to arrive at a final concrete expression for our new <u>covariant derivative</u> (transforms like a tensor).

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a. Linearity: $\nabla(\underbrace{T+S}_{\text{vectors}}) = \nabla T + \nabla S$

Leibniz: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes \nabla S$

Both are satisfied by: $\nabla_\mu V^\nu = \partial_\mu V^\nu + \underbrace{\Gamma^\nu_{\mu\lambda} V^\lambda}$

this is a linear transformation of V,

i.e. $\nabla_0 V^\nu = \partial_0 V^\nu + \underbrace{\Gamma^\nu_{0\lambda} V^\lambda}_{MV}$

$$\nabla_\mu (T^\nu + S^\nu) = \partial_\mu (T^\nu + S^\nu) + \Gamma^\nu_{\mu\lambda}(T^\lambda + S^\lambda)$$
$$= \partial_\mu T^\nu + \Gamma^\nu_{\mu\lambda} T^\lambda + \partial_\mu S^\nu + \Gamma^\nu_{\mu\lambda} S^\lambda$$
$$= \nabla_\mu T^\nu + \nabla_\mu S^\nu$$

$$\nabla_\mu (T^\nu S^\lambda) = \partial_\mu (T^\nu S^\lambda) + \Gamma^\nu_{\mu\alpha} T^\alpha S^\lambda + \Gamma^\lambda_{\mu\beta} T^\nu S^\beta$$
$$= (\partial_\mu T^\nu) S^\lambda + T^\nu \partial_\mu S^\lambda + (\Gamma^\nu_{\mu\alpha} T^\alpha) S^\lambda + T^\nu (\Gamma^\lambda_{\mu\beta} S^\beta)$$
$$= (\nabla T)_\mu{}^\nu S^\lambda + T^\nu (\nabla S)_\mu{}^\lambda$$

b. We want: $\nabla_\mu V^\nu \to \underbrace{\nabla_{\mu'}}_{\text{contains } \Gamma^{\nu'}_{\mu'\lambda'}} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \underbrace{\nabla_\mu}_{\text{contains } \Gamma^\nu_{\mu\lambda}} V^\nu$

We know how everything else transforms (∂_μ, V^ν) so we can deduce:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \Gamma^\nu_{\mu\lambda}}_{\text{tensorial}} - \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}}_{\text{nontensorial (as expected!)}}$$

Connection Transformation Law

So far everything has been in terms of ∇_μ acting on vectors.

To get ∇_μ acting on tensors we consider:

c. $\nabla_\mu (T^\lambda{}_\lambda) = (\nabla T)_\mu{}^\lambda{}_\lambda \Rightarrow \nabla_\mu$ commutes w/ contractions

or

$$\nabla_\mu (T^\nu{}_\lambda \, \delta^\lambda_\nu) = (\nabla_\mu T^\nu{}_\lambda)\delta^\lambda_\nu + T^\nu{}_\lambda \underbrace{(\nabla_\mu \delta^\lambda_\nu)}_{=0} \quad \text{using Leibniz}$$

$$= (\nabla T)_\mu{}^\nu{}_\lambda \, \delta^\lambda_\nu \quad \text{if}$$

$$= (\nabla T)_\mu{}^\lambda{}_\lambda$$

So this implies $\nabla_\mu \delta^\lambda_\nu = 0$

To extend to dual vectors we use:

d. $\nabla_\mu C = \partial_\mu C \Rightarrow \nabla_\mu(\omega_\lambda V^\lambda) = (\nabla_\mu \omega_\lambda)V^\lambda + \omega_\lambda(\nabla_\mu V^\lambda)$

$$= (\partial_\mu \omega_\lambda + \tilde{\Gamma}^\sigma_{\mu\lambda}\omega_\sigma)V^\lambda + \omega_\lambda(\partial_\mu V^\lambda + \Gamma^\lambda_{\mu\nu}V^\nu)$$

$$= (\partial_\mu \omega_\lambda)V^\lambda + \omega_\lambda(\partial_\mu V^\lambda) + \underbrace{\tilde{\Gamma}^\sigma_{\mu\lambda}\omega_\sigma V^\lambda + \Gamma^\lambda_{\mu\nu}\omega_\lambda V^\nu}_{=0}$$

$$= \partial_\mu(\omega_\lambda V^\lambda) \quad \text{if}$$

So we need: $\tilde{\Gamma}^\sigma_{\mu\lambda}\omega_\sigma V^\lambda = -\underbrace{\Gamma^\lambda_{\mu\nu}\omega_\lambda V^\nu}_{\text{switch } \lambda\to\sigma,\ \nu\to\lambda}$

$$\tilde{\Gamma}^\sigma_{\mu\lambda}\omega_\sigma V^\lambda = -\Gamma^\sigma_{\mu\lambda}\omega_\sigma V^\lambda \Rightarrow \tilde{\Gamma}^\sigma_{\mu\lambda} = -\Gamma^\sigma_{\mu\lambda}$$

Or: $\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda$

For general tensors:

$$\boxed{\nabla_\mu T^\alpha{}_\beta = \partial_\mu T^\alpha{}_\beta + \Gamma^\alpha_{\mu\lambda}T^\lambda{}_\beta - \Gamma^\delta_{\mu\beta}T^\alpha{}_\delta} \quad \text{Covariant Derivative of Tensors}$$

With everything we have imposed so far, there are still several possibilities for $\Gamma^\lambda_{\mu\nu}$ that may exist on a given space. The final two criteria specialize to a very particular $\Gamma^\lambda_{\mu\nu}$ called the Christoffel connection.

e. Suppose we had 2 different connections $\Gamma^\lambda_{\mu\nu}$ and $\tilde{\Gamma}^\lambda_{\mu\nu}$.

Using these we can form: $\underbrace{\nabla_\mu V^\lambda - \tilde{\nabla}_\mu V^\lambda}_{\text{this is a tensor!}} = \partial_\mu V^\lambda + \Gamma^\lambda_{\mu\nu} V^\nu - \partial_\mu V^\lambda - \tilde{\Gamma}^\lambda_{\mu\nu} V^\nu$

$$= \underbrace{(\Gamma^\lambda_{\mu\nu} - \tilde{\Gamma}^\lambda_{\mu\nu}) V^\nu}_{\text{so this has to be a tensor!}}$$

$$\Downarrow$$

differences of connections are tensors

So start w/ one connection $\Gamma^\lambda_{\mu\nu}$ and form: $\underbrace{T^\lambda_{\mu\nu}}_{\text{torsion tensor}} \equiv \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = \underbrace{2\Gamma^\lambda_{[\mu\nu]}}_{\text{This is a tensor}}$

But in general: $\underbrace{\Gamma^\lambda_{\mu\nu}}_{\text{not a tensor}} = \underbrace{\tfrac{1}{2}\Gamma^\lambda_{[\mu\nu]}}_{\text{tensor}} + \underbrace{\tfrac{1}{2}\Gamma^\lambda_{(\mu\nu)}}_{\text{not a tensor}}$

So starting w/ a general connection $\Gamma^\lambda_{\mu\nu}$, we can define its torsion-free version $\Gamma^\lambda_{(\mu\nu)}$.

d. Lastly, metric compatibility means: $\nabla_\mu g_{\lambda\rho} = 0$ (the metric is covariantly constant... not constant!)

Along w/ being torsion-free, this condition specifies the Christoffel connection and its form.

Consider:

(1) $\rho\mu\nu$: $\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0$

(2) $\mu\nu\rho$: $\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \underbrace{\Gamma^\lambda_{\mu\rho} g_{\nu\lambda}}_{\Gamma^\lambda_{\rho\mu} g_{\lambda\nu}} = 0$

(3) $\nu\rho\mu$: $\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \underbrace{\Gamma^\lambda_{\nu\rho} g_{\lambda\mu}}_{\Gamma^\lambda_{\rho\nu} g_{\mu\lambda}} - \Gamma^\lambda_{\nu\mu} \underbrace{g_{\rho\lambda}}_{g_{\lambda\rho}} = 0$

Then: $(1)-(2)-(3) = \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0$

$$\Rightarrow \boxed{\Gamma^\sigma_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)}$$ The Christoffel connection!

Note: This is all coordinate dependent, so if we change coordinates $x^\mu \to x^{\mu'}$ then:

$\left(g_{\mu\nu} \to g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}\right.$ which you use to build $\left.\Gamma'\right) = \left(\Gamma^{\sigma'}_{\mu'\nu'} = \frac{\partial x^{\sigma'}}{\partial x^\sigma}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\Gamma^\sigma_{\mu\nu} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial^2 x^{\sigma'}}{\partial x^\mu \partial x^\nu}\right)$

Christoffel Connection

So we have: $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$ where $\Gamma^\nu_{\mu\lambda} = \frac{1}{2} g^{\nu\rho}(\partial_\mu g_{\lambda\rho} + \partial_\lambda g_{\rho\mu} - \partial_\rho g_{\mu\lambda})$

It might suprise you that a) you have used these before

b) Γ's are useful even in flat space

Example: $\mathbb{R}^2$ w/ (r,θ) $\Rightarrow$ $ds^2 = dr^2 + r^2 d\theta^2 \Rightarrow \underbrace{g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}}$

$\Gamma^r_{\theta\theta} = -r$, $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$ all others $= 0$

Then: $\underbrace{\nabla_\mu V^\mu}_{\vec{\nabla}\cdot\vec{V}} = \delta^\mu_\nu \nabla_\mu V^\nu = \delta^\mu_\nu(\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda)$

$= \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$

$= \partial_r V^r + \partial_\theta V^\theta + \Gamma^r_{rr} V^r + \Gamma^r_{r\theta} V^\theta + \Gamma^\theta_{\theta r} V^r + \Gamma^\theta_{\theta\theta} V^\theta$

$= \partial_r V^r + \partial_\theta V^\theta + \frac{1}{r} V^r$

Compared w/ expressions from E&M books: $\vec{\nabla}\cdot\vec{V} = \partial_r V^r + \underbrace{\frac{1}{r}\partial_\theta}_{} V^\theta + \frac{1}{r} V^r$

In reality, the r,θ should be $\hat{r},\hat{\theta}$!!

disagreement w/ our expression

But: E&M books $\Rightarrow$ orthonormal basis, i.e. $\hat{e}_{(r)}\cdot\hat{e}_{(r)} = 1$, $\hat{e}_{(\theta)}\cdot\hat{e}_{(\theta)} = 1$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Our course $\Rightarrow$ coordinate basis, i.e. $e_{(r)}\cdot e_{(r)} = 1$, $e_{(\theta)}\cdot e_{(\theta)} = r^2$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

Then since $V^\mu e_{(\mu)} = V^{\hat{\mu}} \hat{e}_{(\mu)} \Rightarrow V^r = V^{\hat{r}}$, $V^\theta = \frac{1}{r} V^{\hat{\theta}}$ (then they agree!)

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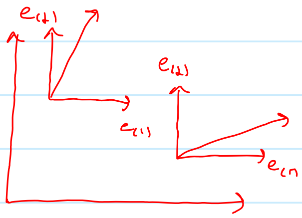
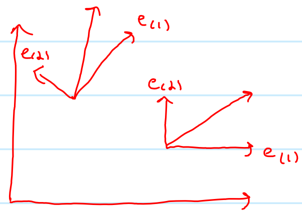
Okay, so what is really going on? $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$ describes how V^ν changes as we move around, but this can happen in 2 ways.

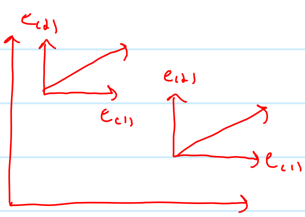


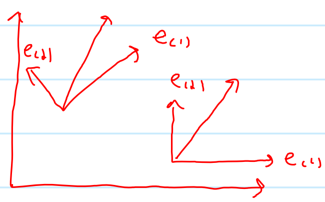


$$\underbrace{\begin{array}{ll} \partial_\mu V^\nu \neq 0 & \qquad \partial_\mu V^\nu = 0 \\ \Gamma^\nu_{\mu\lambda} V^\lambda = 0 & \qquad \Gamma^\nu_{\mu\lambda} V^\lambda \neq 0 \end{array}}$$

In both cases $\nabla_\mu V^\nu \neq 0$ so the vector changes.

If $\nabla_\mu V^\nu = 0$, the V^ν is <u>covariantly</u> <u>constant</u>.





$$\left.\begin{array}{l} \partial_\mu V^\nu = 0 \\ \Gamma^\nu_{\mu\lambda} V^\lambda = 0 \end{array}\right\} \nabla_\mu V^\nu = 0 \qquad \left.\begin{array}{l} \partial_\mu V^\nu \neq 0 \\ \Gamma^\mu_{\mu\lambda} V^\lambda \neq 0 \end{array}\right\} \text{but } \nabla_\mu V^\nu = 0\,!$$

So $\partial_\mu V^\nu$ encodes how the components relative to the basis change, and $\Gamma^\mu_{\nu\lambda}$ tells us how the basis change as we move.